Analysis 1B — Tutorial 5

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# Introduction

Here is the material to accompany the 5th Analysis 1B Tutorial on the 6th March. Alternative formats can be downloaded by clicking the download icon at the top of the page. Please send any comments or corrections to [Christian Jones (caj50)](mailto:caj50@bath.ac.uk). To return to the homepage, click [here](http://caj50.github.io/tutoring.html).

# Lecture Recap

After what was mainly revision last week, we’re moving onto some new stuff again! It turns out there’s still a bit we can say about continuity, especially on compact intervals. Finally, we’re going to look at differentiation, which gives us a way of describing how fast a function changes.

## Inverse Functions

A particularly useful class of functions we may be interested in are known as invertible. These functions provide a way of moving between sets and (and back again) without losing any information about and . Before we talk about them in more detail, it’s worth recalling some definitions:

Definition 1.1 (Injectivity, Surjectivity and Bijectivity)

Let be a function.

* If with , , then is said to be injective.
* If , such that , then is surjective.
* If is both injective and surjective, then it is called bijective.

In words, bijectivity means that for a function , every element in the codomain is mapped to by a unique element in the domain . These bijective functions are said to be invertible, that is, there exists an inverse function such that and produce the identity maps on and respectively.

Now that we have these definitions, we can say something about the continuity of inverse functions:

Theorem 1.1

Let be a non-empty[[1]](#footnote-26) interval, and let be continuous on . Assume that is strictly increasing[[2]](#footnote-27) (or strictly decreasing) on . Then:

* is an interval,
* is bijective, and
* is continuous on .

You’ve seen an example of this theorem in action in the lectures. This is repeated below, as we’re going to use it to prove a powerful result regarding sequences.

Example 1.1

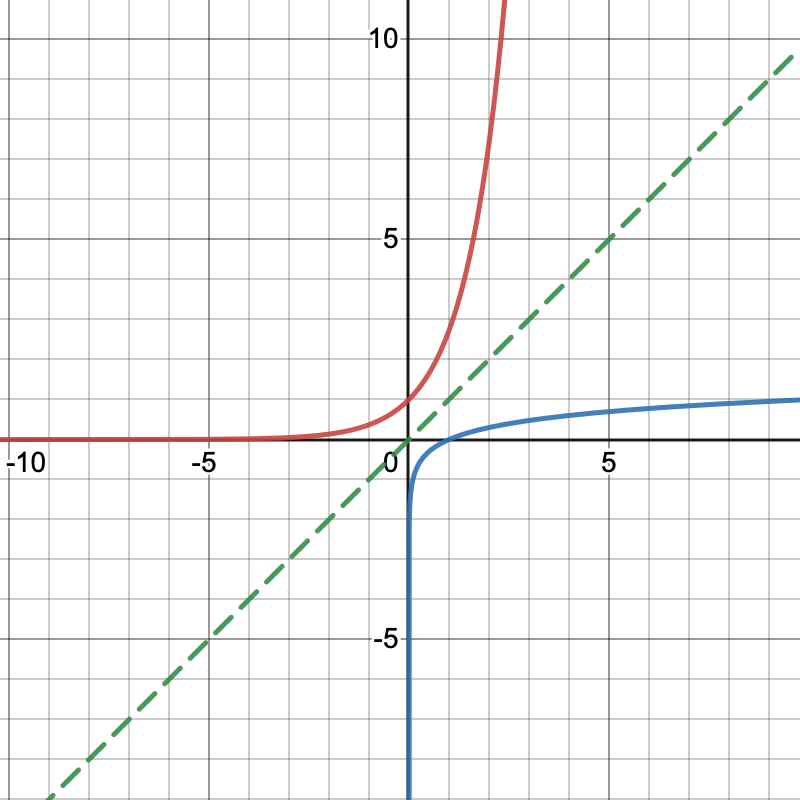
Consider the exponential function defined by

Firstly, note that is a non-empty interval. Now, using some results from Semester 1, we know that

* is continuous and strictly increasing on , and
* .

Therefore, satisfies the hypotheses of the above theorem, and so is a bijection, with continuous inverse. This inverse function is the well-known *natural logarithm* where

We can plot the graphs of (in red), and (in blue) to visually see that Theorem 1.1 works. Also note that to plot the graph of an inverse function, we only need to reflect the graph of the original function through the line (dashed green line).



Now that we have this example, we can easily calculate another large class of sequence limits:

Proposition 1.2

Let and be real sequences such that and . If , and , then

**Proof**

Proof.

Since , then such that ,

In particular, for all .

Now, for ,

So as , we have that as both and are continuous (Example 1.1),

□

## Weierstrass Extremal Theorem

Much like the Intermediate Value Theorem, we can obtain some special continuity results when our functions are defined on compact (i.e. closed and bounded) intervals. One of the main results from this week is stated below:

Theorem 1.3 (Weierstrass Extremal Theorem (WET))

Let with , and let[[3]](#footnote-36) . Then:

1. is bounded:
2. attains its bounds:

* This last point states that if ,
* So in fact, what this theorem tells us is that for a function defined on a compact interval, we have some control on its growth, and we know that the function has a maximum and minimum value! This can be seen pictorally in Figure 1.1.

Figure 1.1: This function f is continuous on [a,b], so by the Weierstrass Extremal Theorem, f is bounded on [a,b]. Also, we see that there exist points p and q in the domain at which f achieves its maximum and minimum values.

Figure 1.1: This function is continuous on , so by the Weierstrass Extremal Theorem, is bounded on . Also, we see that there exist points and in the domain at which achieves its maximum and minimum values.

## Differentiation

While functions are very good at describing physical quantities such as temperature, density or momentum we can usually gain more insight into these variables by studying how fast they change at a given position or time. Mathematically, we study rates of change using derivatives, which again relies on the ideas behind limits!

Definition 1.2 (Derivative)

Let , where is an open set, and let . Then, if such that

we say that is differentiable at , and call the derivative of at .

We can note a few things here:

* Firstly, if this exists, we write it as to make it clear that its a derivative.
* We require to be open, so that we can actually take limits! If, for example, , we could attempt to define the derivative at any point in the interior of , , but we couldn’t define the derivative at or .[[4]](#footnote-41)
* Substituting into the definition gives us an equivalent definition: is differentiable at if there exists such that

One quick result we obtain from this definition is the following:

Proposition 1.4

If a function is differentiable at a point , then it is continuous at .

The contrapositive of this is very useful for ruling functions out: if a function is **not** continuous, it is not differentiable. As a final remark, or warning, **continuity does not imply differentiability**! To see this, think of either at , or look up the [Weierstrass function](https://en.wikipedia.org/wiki/Weierstrass_function).

# Hints

As per usual, here’s where you’ll find the problem sheet hints!

1. This one is largely similar to the one that was covered in tutorials — you just need to be a bit more careful when verifying the hypothesis of the theorem involving inverse functions. When proving bijectivity, you can use results from tutorial question 1 to help too!
   1. The question you’re trying to answer here is does exist?
   2. For , .
   3. This is a bit tricky[[5]](#footnote-47). Consider the case first, and use inertia to show that such that on some interval. For the case , recall that continuity of says that for any , there exists a such that
   4. Using each side of this inequality in turn, the definition of , and part ii), you need to show that for , we have
   5. Combine these inequalities to then prove continuity of .

1. This is so we can talk about surjectivity. [↑](#footnote-ref-26)
2. In other words, for all with . [↑](#footnote-ref-27)
3. Recall that is the set of continuous functions mapping from the set . [↑](#footnote-ref-36)
4. There is nothing stopping us; however, trying to define *left* and *right derivatives* at these points, i.e. we could search for [↑](#footnote-ref-41)
5. Alternatively, you could try and find left and right limits at the point , using (some variations of) a result from Problem Sheet 3. Note that this way involves three main cases: , , or is in . (There’s also a fourth case when and is defined at a single point, but then is automatically continuous.) [↑](#footnote-ref-47)